Lesson 20: Process Characteristics- 2nd Order Lag Process

ET 438a Automatic Control Systems Technology

lesson20et438a.pptx

Learning Objectives

After this series of presentations you will be able to:

- Describe typical 2^{nd} order lag models found in control systems.
- Write mathematical formulas for 2^{nd} order lag process models
- Compute the parameters of this process model.
- Identify the Bode plots of this process model.
- Identify the time response of this process model.

Second-Order Lag Processes

Characteristics: Two energy storage elements
System response determined by three parameters: steady-state gain-G, damping ratio ζ, and resonant frequency, ω_o

Examples: 2 capacitances,
1 mass and 1 spring
1 capacitance and 1 inductance

General Second Order Lag Process Equations

Time domain equation:

$$A_2 \cdot \frac{d^2y}{dt^2} + A_1 \cdot \frac{dy}{dt} + y = G \cdot x$$

Transfer function:

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 \cdot s + A_2 \cdot s^2}$$

Parameters in terms of coefficients A_1 and A_2

$$\omega_0 = \sqrt{\frac{1}{A_2}} \qquad \zeta = \frac{A_1}{2 \cdot \sqrt{A_2}} = \frac{A_1 \cdot \omega_0}{2}$$

Coefficients in terms of Parameters ζ and ω_o

$$A_2 = \frac{1}{{\omega_0}^2} \qquad A_1 = \frac{2 \cdot \zeta}{\omega_0}$$

General Second Order Lag Process Equations

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + A_1 \cdot s + A_2 \cdot s^2} \qquad A_2 = \frac{1}{\omega_0^{\ 2}} \qquad A_1 = \frac{2 \cdot \zeta}{\omega_0}$$

Combine these two equation and simplify

$$\frac{Y(s)}{X(s)} = \frac{G}{1 + \left[\frac{2\cdot\zeta}{\omega_0}\right]\cdot s + \left[\frac{1}{\omega_0^{\ 2}}\right]\cdot s^2}$$

$$\frac{Y(s)}{X(s)} = \left[\frac{\omega_0^{\ 2}}{\omega_0^{\ 2}}\right]\left[\frac{G}{1 + \left[\frac{2\cdot\zeta}{\omega_0}\right]\cdot s + \left[\frac{1}{\omega_0^{\ 2}}\right]\cdot s^2}\right]$$

$$\frac{Y(s)}{X(s)} = \frac{G \cdot \omega_0^{\ 2}}{\omega_0^{\ 2} + 2 \cdot \zeta \cdot \omega_0 \cdot s + s^2}$$

Characteristic Equation: Roots determine system response

Second Order System Responses

Find roots to characteristic equation

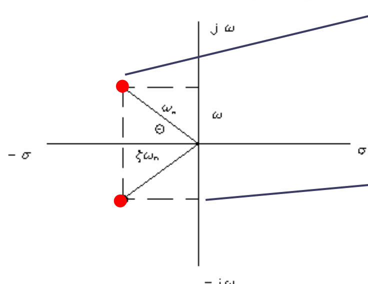
$$\omega_0^{\ 2} + 2 \cdot \zeta \cdot \omega_0 \cdot s + s^2 = 0$$

$$s_1 = -\zeta \cdot \omega_0 + j \cdot \omega_0 \cdot \sqrt{\zeta^2 - 1} = \sigma_1 + j \cdot \omega_1$$

$$s_2 = -\zeta \cdot \omega_0 - j \cdot \omega_0 \cdot \sqrt{\zeta^2 - 1} = \sigma_2 + j \cdot \omega_2$$

Two poles at these locations

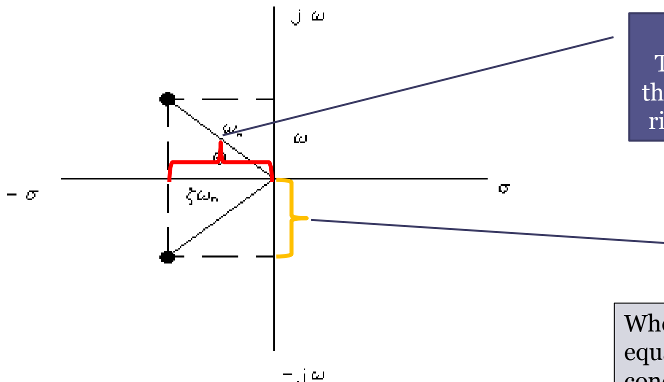
Plot these roots on complex plane



As poles near imaginary axis system become more oscillatory

If ζ=0, damping is zero and system will oscillate at $\omega = \omega_0$

Second Order System Responses



$\omega_0\zeta=\sigma$
This controls the exponential rise and decay

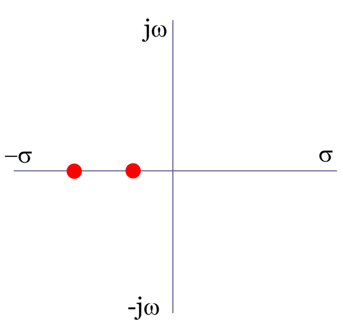
$\omega_0 \cdot \sqrt{\zeta^2 - 1}$

When $0<\zeta<1$ above equation determines conditional frequency- a damped sinusoid

Second Order System Responses

Roots of quadratic formula can have three possible forms

1) real - distinct
2) real - identical
3) imaginary - conjugate pairs



Location of roots is controlled by the values of ζ and ω_0.

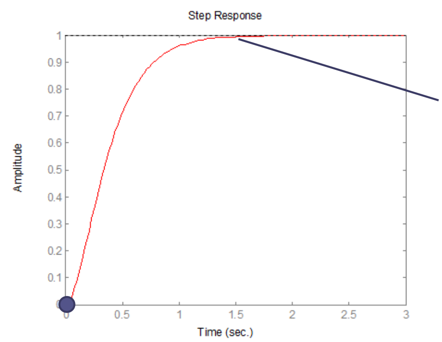
If natural frequency is constant them damping controls system response

Second Order System Responses

Damping coefficient value and system responses

$\zeta = 1$ **- critically damped system.** Reaches the final value the fastest without having any overshoot. Roots are equal and real.

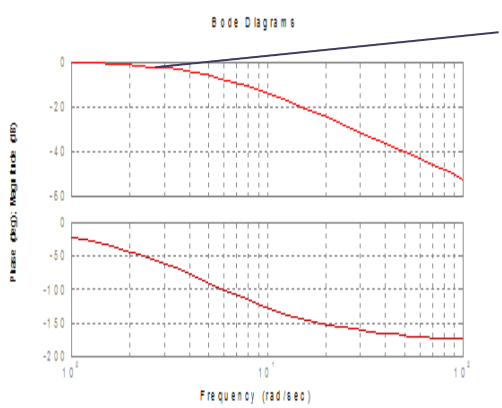
Time response



Final value in approx. 1.4 seconds No Overshoot.

Second Order System Responses-Critically Damped

Bode plot of Critically damped system

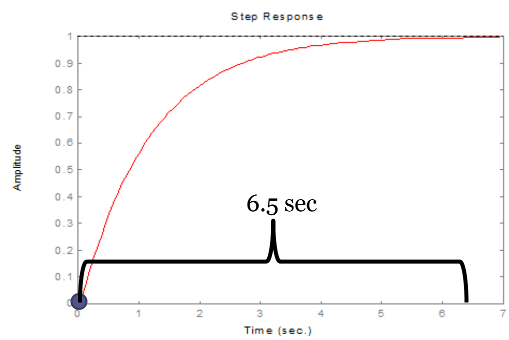


Two poles at this point -3 dB from max. gain

Response similar to lag process

Second Order Response-Over Damped System

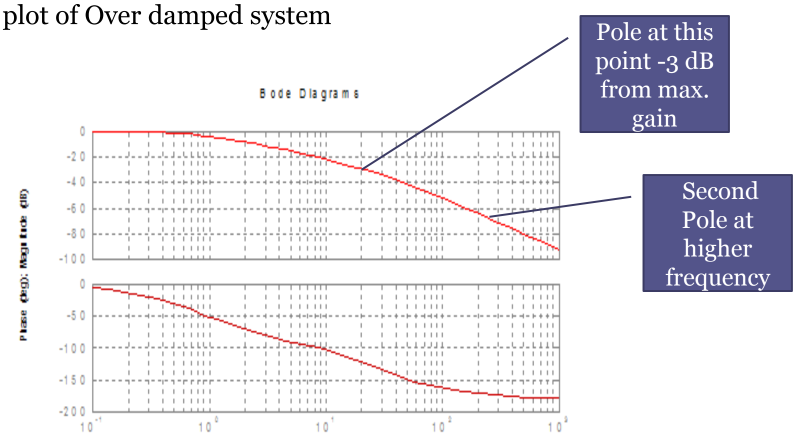
$\zeta > 1$ **- over damped system**. Reaches the final value slowly but with no overshoot. More damping, slower response to final value. Roots are real but not equal.



Compared to the critically damped case, the response time is slower. Approx. 6.5 sec to get to final value vs 1.4 sec

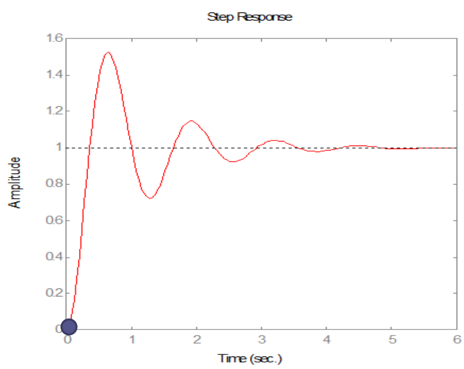
Over Damped System Frequency Response

Bode plot of Over damped system



Second Order Response-Under Damped System

$\zeta < 1$ - under damped system . Reaches the final value fast but with overshoot. Less damping more overshoot. Roots are conjugate pairs.

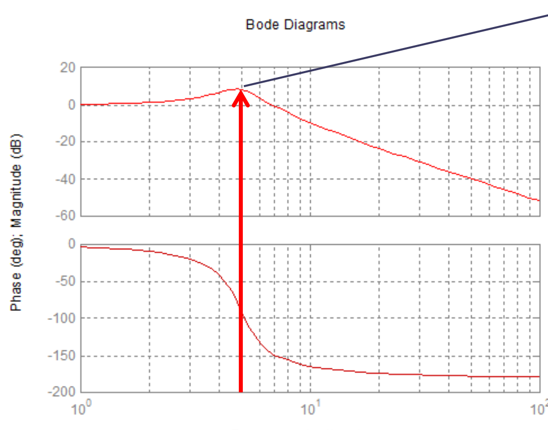


Overshoots to 1.5

Settling time is time Required to reach % of final value Approximately 4.5 sec

Under Damped System Frequency Response

Bode plot of under damped system



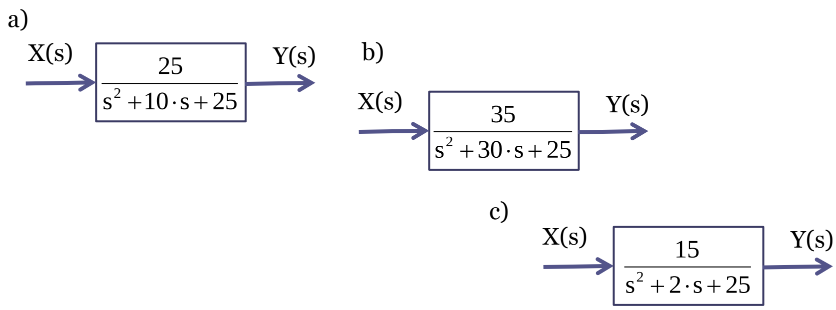
Resonant Peak at 5 rad/sec

Natural oscillating frequency of system

90 degree phase shift at resonant frequency

Determining System Parameters From Characteristic Equation

Example 20-1: The block diagrams shown below represent three second order systems. Use the characteristic equations of each transfer function to determine the values of ω_0 and ζ for each and determine if each system is over, under or critically damped.



Example 20-1 Solution (1)

a) Equate the general characteristic equation with that of the transfer function.

$$s^2+10\cdot s+25=s^2+2\cdot\zeta\cdot\omega_0\cdot s+\omega_0^{\,2}$$

Equate Coefficients $25=\omega_0^{\,2}$ $\sqrt{25}=\omega_0=5$ Ans

$$10=2\cdot\zeta\cdot\omega_0 \quad \text{so} \quad 10=2\cdot\zeta\cdot 5$$

$\zeta=1$ Ans Critical damped

b) $\frac{Y(s)}{X(s)}=\frac{35}{s^2+30\cdot s+25}$

$$s^2+30\cdot s+25=s^2+2\cdot\zeta\cdot\omega_0\cdot s+\omega_0^{\,2}$$

$25=\omega_0^{\,2}$ $\sqrt{25}=\omega_0=5$ Ans

$$30=2\cdot\zeta\cdot\omega_0 \quad \text{so} \quad 30=2\cdot\zeta\cdot 5$$

$\zeta=3$ Ans Over damped

Example 20-1 Solution (2)

c) $\frac{Y(s)}{X(s)} = \frac{15}{s^2+2\cdot s+25}$

$$s^2+2\cdot s+25 = s^2+2\cdot\zeta\cdot\omega_0\cdot s+\omega_0^{\,2}$$

$$25=\omega_0^{\,2} \quad \sqrt{25}=\omega_0=5$$ Ans

$$2=2\cdot\zeta\cdot\omega_0 \quad \text{so} \quad 2=2\cdot\zeta\cdot 5$$

$$\zeta=0.2$$ Ans Under damped

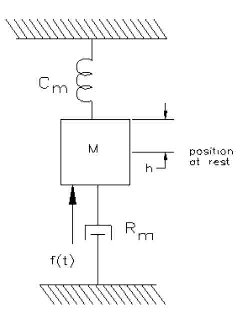
Note: The numerator of the transfer functions does not affect the response of the system. It determines the maximum output.

As damping increases, one of the system poles becomes more dominant. The dominant pole controls system response.

Second Order Mechanical System

Example 20-2: The mechanical system shown below is at rest with an initial height of h(0)=0. An external input force, f(t) disturbs the system. The system output , h(t) is the centerline position of the mass. The system parameters are:

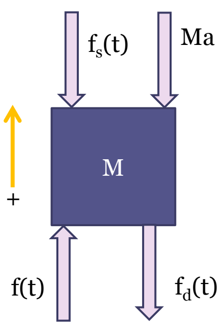
C_m = spring capacitance (Inverse of spring constant, K) =0.001 m/N
R_m = resistance due to viscous friction (B)= 20 N-s/m
M = mass = 10 Kg

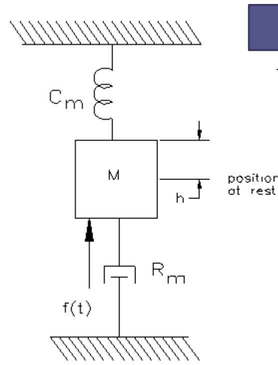


Find: a) coefficients of the second order system equation
b) system transfer function
c) resonant frequency
d) damping ratio
e) if system is over, under or critically damped

Example 20-2 Solution (1)

a) Sum the external forces. From Newton's Law F=Ma





Sum forces

Where:

$$f_s(t) = \frac{h(t)}{C_m}$$

$$f_d(t) = R_m\left(\frac{dh(t)}{dt}\right)$$

$$Ma = M \cdot \frac{d^2h(t)}{dt^2}$$

Example 20-2 Solution (2)

$$f(t) - \frac{h(t)}{C_m} - R_m\left(\frac{dh(t)}{dt}\right) - M \cdot \frac{d^2h(t)}{dt^2} = 0$$

Add terms to both sides

$$f(t) = M \cdot \frac{d^2h(t)}{dt^2} + R_m\left(\frac{dh(t)}{dt}\right) + \frac{h(t)}{C_m}$$

$$C_m \cdot f(t) = C_m \cdot M \cdot \frac{d^2h(t)}{dt^2} + C_m \cdot R_m\left(\frac{dh(t)}{dt}\right) + h(t)$$

$$G \cdot x(t) = A_2 \cdot \frac{d^2y(t)}{dt^2} + A_1 \cdot \left(\frac{dy(t)}{dt}\right) + y(t)$$

General form of 2nd order system

Ans a

$$A_2 = C_m \cdot M = (0.001\ m/N)(10\ Kg) = 0.01\ sec^2$$

$$A_1 = C_m \cdot R_m = (0.001\ m/N)(20\ N\text{-}sec/m) = 0.02\ sec$$

$$G = C_m = 0.001\ m/N$$

Example 20-2 Solution (2)

b) Find transfer function

$$0.01\cdot\frac{d^2h(t)}{dt^2}+0.02\left(\frac{dh(t)}{dt}\right)+h(t)=0.001\cdot f(t)$$

$s^2H(s)$ — $sH(s)$ — $F(s)$

Take Laplace transform of above equation and factor out common term

$$0.01\cdot s^2\cdot H(s)+0.02\cdot s\cdot H(s)+H(s)=0.001\cdot F(s)$$

$$H(s)=\frac{0.001\cdot F(s)}{(0.01\cdot s^2+0.02\cdot s+1)}$$

$$\frac{H(s)}{F(s)}=\frac{0.001}{(0.01\cdot s^2+0.02\cdot s+1)}$$

Ans

Example 20-2 Solution (3)

c) Find the resonant frequency ω_o

$$\omega_0=\sqrt{\frac{1}{A_2}}=\sqrt{\frac{1}{0.01\,\text{sec}^2}}=10\text{ rad/sec}$$

Ans

d) Find the damping ratio ζ

$$\zeta=\frac{A_1}{2\cdot\sqrt{A_2}}=\frac{0.02\text{ sec}}{2\cdot\sqrt{0.01\,\text{sec}^2}}=0.1$$

Ans

e) The damping ratio is $\zeta<1$, so the system is under damped. Applying external force causes the mass center to oscillate before coming to rest at a new position.

End Lesson 20: Process Characteristics-2nd Order Process

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